Numerical bifurcation analysis of delay equations using software for ODEs: the pseudospectral discretization approach

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Acknowledgements

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Alessia Andò
Simone De Reggi
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Nonlinear delay equation $\rightsquigarrow$

ODE approximation and software for ODE e.g. MatCont

$\rightsquigarrow$ bifurcation analyses

Breda, Diekmann, Gyllenberg, S., Vermiglio, *SIAM J Appl Dyn Syst*, 2016 (and following)

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Delay equations from population dynamics

Renewal equation for population birth rate

$$b(t) = \int_{a_{\text{repr}}}^{a_{\max}} \overbrace{\beta(a)}^{\text{fertility}} \, \overbrace{\mathcal{F}(a)\, b(t-a)}^{\text{ind of age } a} \, da$$

often coupled with a delay-differential equation for the environmental variable (substrate, prey,...)

$$\frac{dS}{dt}(t) = \underbrace{f(S(t))}_{\text{consumer-free}} - \int_0^{a_{\max}} \overbrace{\gamma(a)}^{\text{consumption}} \, \overbrace{\mathcal{F}(a)\, b(t-a)}^{\text{ind of age } a} \, da$$

Delay equation: a rule for extending a function given its past

Let $\tau > 0$ be the maximal delay. Given a function x, the history function is

$$x_t \colon [-\tau, 0] \to \mathbb{R}^d$$
$$x_t(\theta) = x(t+\theta), \quad \theta \in [-\tau, 0]$$

renewal (RE):	$x(t) = F(x_t)$	$F \colon X \to \mathbb{R}^d$ $X = L^1([-\tau, 0], \mathbb{R}^d)$
differential (DDE):	$\dot{x}(t) = F(x_t)$	$F \colon X \to \mathbb{R}^d$ $X = C([-\tau, 0], \mathbb{R}^d)$
coupled systems:	$\begin{cases} x(t) = F(x_t, y_t) \\ \dot{y}(t) = G(x_t, y_t) \end{cases}$	$F, G \colon X \times Y \to \mathbb{R}^d$ $X = L^1,\ Y = C$

The abstract equation

$$\text{(RE)} \quad x(t) = F(x_t), \qquad\qquad \text{(DDE)} \quad \dot{x}(t) = F(x_t)$$

Once provided with an initial condition ψ on $[-\tau, 0]$, the corresponding initial value problem is equivalent to an Abstract Cauchy Problem for the history function $v(t) = x_t \in X$

$$\begin{cases} \dot{v}(t) = \mathcal{A}(v(t)) & t \geq 0 \\ v(0) = \psi \end{cases}$$

where $\mathcal{A}$ is the infinitesimal generator of the family of solution operators, $\{T(t)\}_{t\geq 0}$,

$$\mathcal{A}(\psi) = \psi', \quad \psi \in D(\mathcal{A})$$
$$D(\mathcal{A}) = \{\psi \in X \text{ s.t. } \psi' \in X \text{ and a “rule for extension”}\}$$

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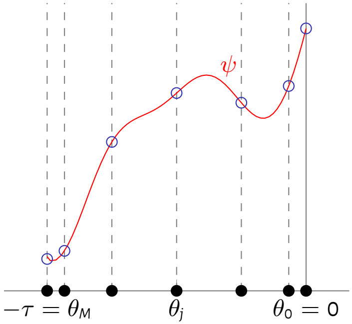
Rule for extension:

- for DDE: $\psi'(0) = F(\psi)$, with $X = C([-\tau, 0])$
- for RE: $\psi(0) = F(\psi)$, with $X = L^1([-\tau, 0])$

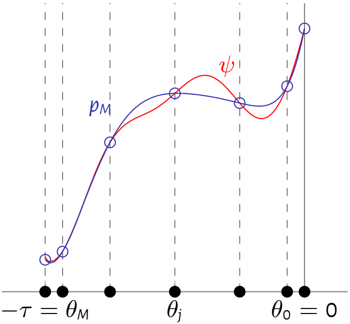
Discretization via pseudospectral methods



Mesh of $M + 1$ nodes in $[-\tau, 0]$

$$-\tau = \theta_M < \cdots < \theta_0 = 0$$

Discretization via pseudospectral methods



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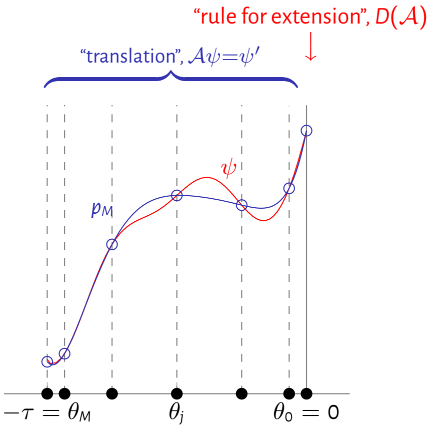
function in Y $\approx$ polynomial of degree M

$$\psi(\theta) \approx p_M(\theta) = \sum_{j=0}^{M} \ell_j(\theta)\psi(\theta_j)$$

with $\ell_j(\theta)$ Lagrange polynomials:

$$\ell_j(\theta) = \prod_{k \neq j} \frac{\theta - \theta_k}{\theta_j - \theta_k}$$

Discretization via pseudospectral methods



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Differentiation

Given $v(t) = x_t$, we consider

$$x_M(t) \approx v(t)(\theta_0) = x(t+\theta_0) = x(t)$$
$$V_{M,j}(t) \approx v(t)(\theta_j) = x(t+\theta_j), \qquad j = 1, \dots, M$$

and let $P_M \colon \mathbb{R} \times \mathbb{R}^M \to C([-\tau, 0])$ be the interpolation operator:

$$P_M(x_M, V_M) = \ell_0(\cdot)x_M + \sum_{j=1}^{M} \ell_j(\cdot)V_{M,j}$$

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$$P_M(x_M, V_M) = \ell_0(\cdot)x_M + \sum_{j=1}^{M} \ell_j(\cdot)V_{M,j}$$

then we approximate $\mathcal{A}\psi = \psi'$ by pseudospectral differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}\theta} P_M(x_M, V_M)(\theta_k) = \ell_0'(\theta_k)x_M + \sum_{j=1}^{M} \ell_j'(\theta_k)V_{M,j}, \qquad k = 1, \dots, M$$

so we obtain the set of linear equations

$$\dot{V}_M = (d_M \mid D_M) \begin{pmatrix} x_M \\ V_M \end{pmatrix}$$

with:

$$(d_M \mid D_M) := \left(\begin{array}{c|ccc} \ell_0'(\theta_1) & \ell_1'(\theta_1) & \dots & \ell_M'(\theta_1) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_0'(\theta_M) & \ell_1'(\theta_M) & \dots & \ell_M'(\theta_M) \end{array}\right) \qquad d_M \in \mathbb{R}^M,\ D_M \in \mathbb{R}^{M\times M}$$

Approximating the rule for extension

The equation for $x_M(t)$ is obtained from the rule for extension (that specifies the domain of $\mathcal{A}$).

- for DDE, the rule for extension is differential ($\psi'(0) = F(\psi)$), and collocation gives:

$$\dot{x}_M = F(p_M), \qquad p_M := P_M(x_M, V_M)$$
$$\dot{V}_M = d_M x_M + D_M V_M$$

$\rightarrow$ approximating system of $M + 1$ ODE

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- for RE, the rule for extension is not differentiated ($\psi(0) = F(\psi)$), and collocation gives:

$$x_M = F(p_M), \qquad p_M := P_M(x_M, V_M)$$
$$\dot{V}_M = d_M x_M + D_M V_M$$

$\rightarrow$ approximating system of M ODE plus algebraic condition for x_M

A closer look at the approximating systems

Approach in Breda et al, *SIADS*, 2016

$$\text{(DDE)} \begin{cases} \dot{x}_M = F(P_M(x_M, V_M)) \\ \dot{V}_M = d_M x_M + D_M V_M \end{cases} \qquad \text{(RE)} \begin{cases} x_M = F(P_M(x_M, V_M)) \\ \dot{V}_M = d_M x_M + D_M V_M \end{cases}$$

- linear differentiation part independent of the specific equation
- the right-hand side F appears only in the equation for x_M
- the ODE formulation can be studied with available software for ODEs
- REs:
  - require to solve an algebraic condition at every step (computationally expensive):

    $$x_M(t) = h(x_M(t), V_M(t))$$

  - however, in structured population models F is often linear:

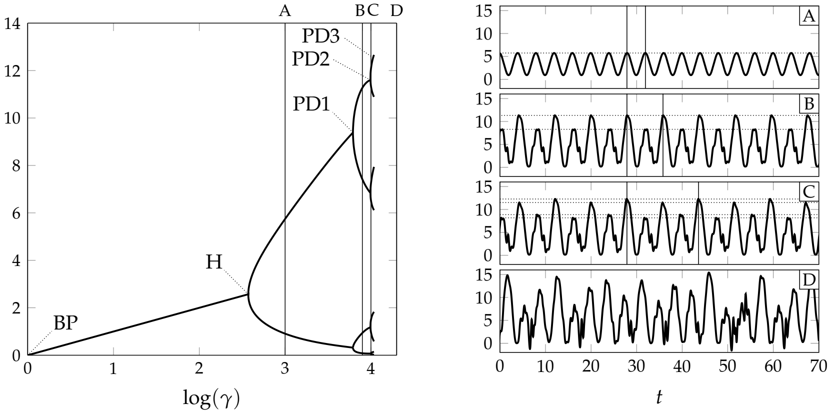
    $$x_M(t) = (I - F\ell_0)\, F\Big(\sum_{j=1}^{M} V_{M,j}(t)\ell_j\Big) = \tilde{h}(V_M(t))$$

  - approximating L^1 functions with polynomials seems unnatural

Example: nonlinear renewal equation

$$b(t) = \frac{\gamma}{2} \int_1^\tau b(t-s)\, \mathrm{e}^{-b(t-s)}\, \mathrm{d}s, \quad t \geq 0,$$

Bifurcation analysis and periodic orbits obtained with Matcont, $\tau = 3$



Breda, Diekmann, Liessi, S., *Electr. J. Qual. Th. Diff. Equ.*, 2016

The abstract equation via suitable injection

$$\text{(RE)} \quad x(t) = F(x_t), \qquad\qquad \text{(DDE)} \quad \dot{x}(t) = F(x_t)$$

Via a suitable injection $j\colon X \to Y$, the delay equation can be reformulated as a semilinear ADE for $v(t) = jx_t \in Y$ as

$$\dot{v}(t) = \mathcal{A}v(t) + \mathcal{F}(v(t)), \quad t \geq 0$$

where

- $\mathcal{A}$ is the infinitesimal generator associated with the trivial problem ($F \equiv 0$)
- $\mathcal{F}$ is a nonlinear perturbation that captures the action of F

Diekmann et al., *SIAM J Math Anal*, 2008
Diekmann and Verduyn Lunel, *J Diff Equ*, 2021

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For DDEs, one can use the sun-star theory to map $j\colon C([-\tau, 0]) \to \mathbb{R} \times L^{\infty}([-\tau, 0])$, obtaining the same discretization as before

Diekmann et al., *SIAM J Math Anal*, 2008
Diekmann, S., Vermiglio, *DCDS*, 2020

For REs, we use a suitable integral mapping j

Renewal equations

$$x(t) = F(x_t) \qquad \Leftrightarrow \qquad \dot{v}(t) = \mathcal{A}v(t) + \mathcal{F}(v(t))$$

Consider the mapping

$$j \colon L^1 \to AC \subset NBV$$
$$\varphi \mapsto -\int_{\cdot}^{0} \varphi(s)\,\mathrm{d}s$$

then $\mathcal{A}$ becomes

$$\mathcal{A}\psi = \psi', \qquad \psi \in D(\mathcal{A})$$
$$D(\mathcal{A}) = \{\psi \in AC \colon \psi' \in NBV,\ \psi(0) = 0\}.$$

and

$$\mathcal{F}(\psi) = qF(\mathcal{A}\psi), \qquad \text{for} \quad q(\theta) := \begin{cases} 0, & \theta = 0, \\ -1, & \theta \in [-\tau, 0) \end{cases}$$

Diekmann, Verduyn Lunel, *J Diff Equ*, 2021

Renewal equations: pseudospectral approximation

Approach in Scarabel et al, *JCAM*, 2021

$$x(t) = F(x_t) \qquad \Leftrightarrow \qquad \dot{v}(t) = \mathcal{A}v(t) + \mathcal{F}(v(t))$$

We approximate $v(t) = jx_t$ with (note that $v(t)(\theta_0) = 0$)

$$P_M(0, V_M) = \sum_{j=1}^{M} V_{M,j}\ell_j(\cdot), \qquad V_{M,j}(t) \approx v(t)(\theta_j), \quad j = 1, \dots, M$$

Using collocation, we find an M-dimensional system for $V_M \in \mathbb{R}^M$:

$$\dot{V}_M = D_M V_M - F(0, D_M V_M)$$

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- $x_t \in L^1 \Rightarrow v(t) \in AC$
- algebraic condition replaced by nonlinear perturbation
- we do not need to solve a nonlinear equation, but only evaluate the function F
- we can recover the solution of the RE by

$$x(t) \approx F(P_M(0, D_M V_M(t)))$$

Comparison of computation times

$$b(t) = \frac{\gamma}{2} \int_1^{\tau} b(t-s)\, \mathrm{e}^{-b(t-s)}\, \mathrm{d}s, \quad t \geq 0,$$

Computation time (seconds) for a 100-point continuation of the branch of positive equilibria and periodic solutions.

Equilibrium				**Periodic solutions**			
M	[SIADS 2016]	[JCAM 2021]	Ratio	M	[SIADS 2016]	[JCAM 2021]	Ratio
15	15.65	1.27	12.32	15	2704	198	13.66
16	16.60	1.40	11.86	16	2889	259	11.15
17	17.62	1.52	11.59	17	3066	298	10.29
18	18.75	1.66	11.30	18	3265	301	10.85
19	19.79	1.77	11.18	19	3437	302	11.38
20	21.07	1.91	11.03	20	3659	344	10.64

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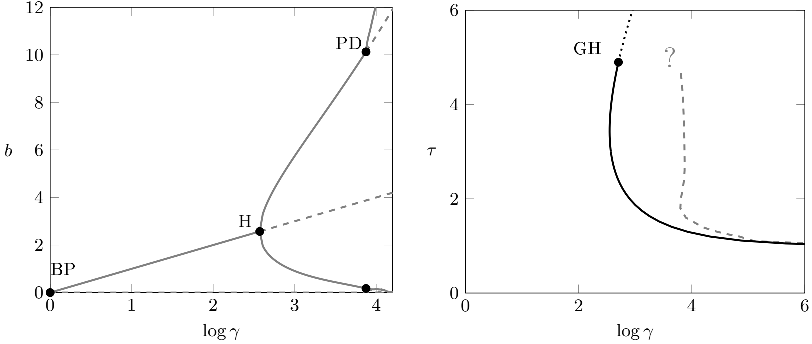
Numerical bifurcation analysis

$$b(t) = \frac{\gamma}{2} \int_1^\tau b(t-s)\, \mathrm{e}^{-b(t-s)}\, \mathrm{d}s, \quad t \geq 0,$$

Improved computation times allowed us to study the two-parameter plane



Performed with Matcont, $M = 20$, $\tau = 3$.

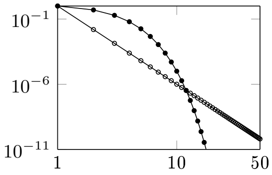
Approximation of equilibria and their stability

- one-to-one correspondence of equilibria $\bar{y} \leftrightarrow (\bar{y}, \cdots, \bar{y})$
- $\tau < \infty$ and Chebyshev nodes: spectral accuracy for characteristic roots, i.e., faster than $O(M^{-k})$ for all k: M =10–20 typically gives satisfactory accuracy

Breda, Maset, Vermiglio, *SIAM J. Sci. Comput.*, 2005
Breda, Getto, Sánchez Sanz, Vermiglio, *SIAM J. Sci. Comput.*, 2015
Breda, Diekmann, Gyllenberg, S., Vermiglio, *SIAM J. Appl. Dyn. Syst.*, 2016
Diekmann, Scarabel, Vermiglio, *J Comp Appl Math*, 2021

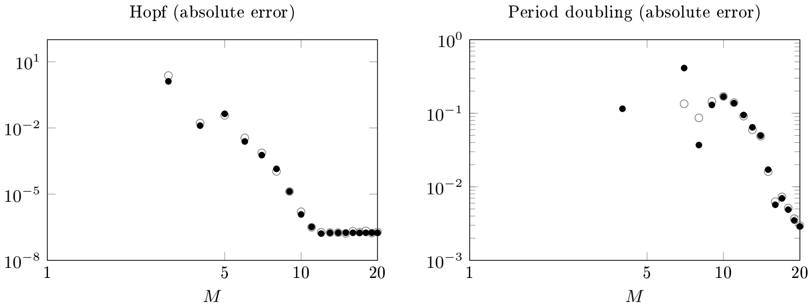
- (DDE) Hopf bifurcation and its direction also approximated with spectral accuracy

de Wolff, S., Verduyn-Lunel, Diekmann, *SIAM J Appl Dyn Syst*, 2020



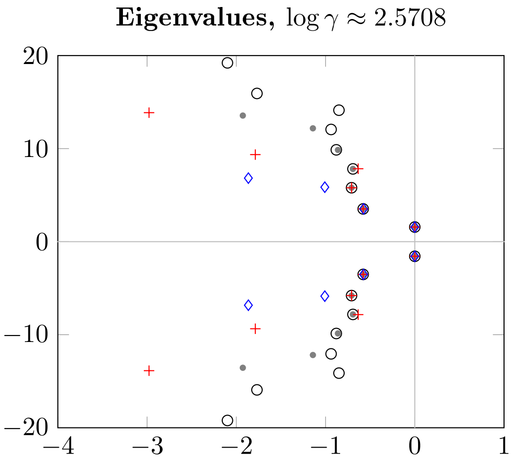
Errors varying M

Errors in approximated Hopf and period doubling bifurcation points, increasing M.



Methods: inversion algebraic condition (SIADS 2016, ○) and integration (JCAM 2021, ●).

Approximated eigenvalues



$M = 10$ ($\diamond$), $M = 15$ ($+$), $M = 20$ ($\bullet$), $M = 25$ ($\circ$). Larger (in modulus) eigenvalues to the left are not visible in the picture.

A unified framework for DDE and RE

Approximating ODE:

$$\dot{U}_M(t) = A_M U_M(t) + F_M(U_M(t))$$

	differential (DDE)	renewal (RE)
equation	$\dot{x}(t) = F(x_t)$	$x(t) = F(x_t)$
state vector	$U_M \in \mathbb{R}^{M+1}$	$U_M \in \mathbb{R}^M$
operators	$A_M, F_M : \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$	$A_M, F_M : \mathbb{R}^M \to \mathbb{R}^M$
A_M	$A_M = \begin{pmatrix} 0 & 0 \\ d_M & D_M \end{pmatrix}$	$A_M = D_M$
F_M	$qF(P_M(U_M))$	$qF(P_M(0, D_M U_M(t)))$
	$q = (1, 0, \ldots, 0)^T \in \mathbb{R}^{M+1}$	$q = (-1, \ldots, -1)^T \in \mathbb{R}^M$
approximation	$x(t) \approx U_{M,0}(t)$	$x(t) \approx F(P_M(0, D_M U_M(t)))$

Concluding remarks

- flexible and general (DDE, RE, ∞, discrete & distributed, state-dependent delay)
- easy to implement
- low dimensional
- exploits pre-existing software for ODE
- no need to linearize

Work in progress

- convergence of stability with infinite delay (using weight $e^{\rho\theta}$)
- structured population models with two structures (PDE)
- convergence of solution operators of IVP
- incorporation with MatCont $\rightarrow$ next talk!

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Thank you!